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About Limit Matrices of Finite-State Markov Chains

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ABSTRACT

By means of the concept of group inverse of a matrix we study limiting properties of a collection of stochastic matrices $\{P_\epsilon, \epsilon \in [0, 1]\}$, where $\forall \varepsilon \in [0, 1]$, $P_\varepsilon \in \mathbb{R}^{n \times n}$ and where $P_0 = \lim_{\varepsilon \to 0} P_\varepsilon$. © *Elsevier Science Inc., 1997*

INTRODUCTION

In the sequel $n \in \mathbb{N}$ is a fixed natural number and all matrices are elements of $\mathbb{R}^{n \times n}$ unless explicitly indicated otherwise.

Let P be a stochastic matrix. Then it is well known that

$$\overline{P} := \lim_{t \to \infty} \frac{I + P + \cdots + P^{t-1}}{t}$$

exists. Under those circumstances one has $\overline{P} = \lim_{t \to \infty} P^t$.

Let M^* be the group inverse of $M := I - P$. Then $\overline{P} = I - MM^*$. Let now $\{P_\varepsilon, \varepsilon \in [0, 1]\}$ be a collection of stochastic matrices with $P_0 = \lim_{\varepsilon \to 0} P_\varepsilon$. We will study the behavior of $\{\overline{P}_\varepsilon, \varepsilon \in [0, 1]\}$ by means of the concept of group inverses, as elaborated in a paper by Robert [1] and applied to Markov chains by Meyer in [2]. In order to make this note also readable for

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the novice in group inverses, we give all the necessary definitions and theorems.

GROUP INVERSE OF A MATRIX

We denote by $\mathscr{R}(M)$ the *range* of a matrix M, and by $\mathscr{N}(M)$ the *null space* of M.

LEMMA 1 (See [1]). *For all $y \in \mathbb{R}^n$ there exists a unique $x \in \mathscr{R}(M)$ with $Mx - y \in \mathscr{N}(M)$ if and only if $\mathbb{R}^n = \mathscr{R}(M) \oplus \mathscr{N}(M)$, i.e., $\mathbb{R}^n$ is the direct sum of $\mathscr{R}(M)$ and $\mathscr{N}(M)$.*

DEFINITION. Let M be such that $\mathbb{R}^n = \mathscr{R}(M) \oplus \mathscr{N}(M)$. Then M^*: $\mathbb{R}^n \to \mathbb{R}^n$, the *group inverse* of M, is defined as follows:

$$M^* y := x \in \mathscr{R}(M) \qquad \text{with} \qquad Mx - y \in \mathscr{N}(M).$$

LEMMA 2 (See [1]). *Let $M := I - P$ with P stochastic. Then $\mathbb{R}^n = \mathscr{R}(M) \oplus \mathscr{N}(M)$.*

LEMMA 3 (See [1]). *Let M be such that $\mathbb{R}^n = \mathscr{R}(M) \oplus \mathscr{N}(M)$. Then $\mathscr{R}(M^2) = \mathscr{R}(M)$ and $\mathscr{N}(M^2) = \mathscr{N}(M)$.*

MAIN RESULTS

In the sequel we will apply the foregoing in the following context: We are given a collection of stochastic matrices $\{P_\varepsilon, \varepsilon \in [0, 1]\}$ with the following property:

$$\lim_{\varepsilon \to 0} P_\varepsilon = P_0.$$

THEOREM. *Let $p \in \mathbb{N}$, $p \leqslant n$, be such that for some $\bar{\varepsilon} \in [0, 1]$ we have $\dim \mathscr{N}(I - P_\varepsilon) = p$ $\forall \varepsilon \in [0, \tilde{\varepsilon}]$. Then, with $M_\varepsilon = I - P_\varepsilon$, we have $\lim_{\varepsilon \to 0} M_\varepsilon^* = M_0^*$.*

An important ingredient in the proof of this theorem is the following lemma.

LEMMA 4. *Let* $x_\varepsilon \in \mathscr{R}(\mathscr{M}_\varepsilon)$ *and* $x_0 = \lim_{\varepsilon \to 0} x_\varepsilon$. *Then* $x_0 \in \mathscr{R}(\mathscr{M}_0)$. *Here* $\{M_\varepsilon\}$ *is as in the Theorem.*

Proof. We have $\dim \mathscr{R}(M_\varepsilon) + \dim \mathscr{N}(M_\varepsilon) = n\ \forall \varepsilon \in [0, \bar{\varepsilon}]$, where $\bar{\varepsilon}$ is as in the statement of the Theorem. Hence $\dim \mathscr{R}(M_\varepsilon) = n - p\ \forall \varepsilon \in [0, \bar{\varepsilon}]$. But this implies that for ε small enough, possibly after some permutation of the columns, $M_\varepsilon = [M_\varepsilon^1, M_\varepsilon^2]$ with rank $M_\varepsilon^1 = n - p$. So for ε small enough, $\varepsilon > 0$, we have $x_\varepsilon = M_\varepsilon^1 y_\varepsilon$.

Assume now, to the contrary, that $\{y_\varepsilon\}$ contains a diverging sequence. Then we find $0 = M_0^1 \hat{y}$ with $\|\hat{y}\| = 1$, but this contradicts the independence of the columns of M_0^1. Hence we may assume that $\lim_{\varepsilon \to 0} y_\varepsilon = y_0$; hence $x_0 = M_0^1 y_0$, and we are done with the proof. ∎

We continue with the proof of the Theorem.

PROOF OF THE THEOREM

We know that M_ε^* exists for all $\varepsilon \in [0, 1]$. In the sequel we will prove that $M_\varepsilon^* y$ converges to $M_0^* y$, when $\varepsilon \to 0$, where $y \in \mathbb{R}^n$ is arbitrary.

By definition we know $M_\varepsilon x_\varepsilon - y = r_\varepsilon$ with $M_\varepsilon r_\varepsilon = 0$ and with $x_\varepsilon \in \mathscr{R}(\mathscr{M}_\varepsilon)$, or $M_\varepsilon^* y = x_\varepsilon$.

Assume to the contrary that $\{x_\varepsilon\}$ contains a divergent subsequence. We know that $\|r_\varepsilon\| = \|M_\varepsilon x_\varepsilon - y\| \leqslant \|M_\varepsilon x_\varepsilon\| + \|y\| \leqslant \|M_\varepsilon\| \|x_\varepsilon\| + \|y\|$, where $\|x\|$ is the Euclidean norm of x and $\|M_\varepsilon\|$ is the induced norm of M_ε. So assume $\|x_\varepsilon\| \to \infty$, and consider $\hat{r}_\varepsilon := r_\varepsilon / \|x_\varepsilon\|$ and $\hat{x}_\varepsilon := x_\varepsilon / \|x_\varepsilon\|$. Without loss of generality we may assume that $\lim_{\varepsilon \to 0} \hat{r}_\varepsilon = \hat{r}_0$ and $\lim_{\varepsilon \to 0} \hat{x}_\varepsilon = \hat{x}_0$.

So we find $M_0 \hat{x} = \hat{r}$ with $\|\hat{x}\| = 1$ and $M_0 \hat{r} = 0$. Therefore $M_0^2 \hat{x} = 0$, and therefore, because of Lemma 3, $\hat{x} \in \mathscr{N}(M_0)$. We also have $x_\varepsilon \in \mathscr{R}(M_\varepsilon)$; hence $\hat{x}_\varepsilon \mathscr{R}(M_\varepsilon)$ and, because of Lemma 4, we find $\hat{x} \in \mathscr{R}(M_0)$. As $\mathscr{N}(M_0) \oplus \mathscr{R}(M_0) = \mathbb{R}^n$, we arrive at a contradiction, and therefore $\{x_\varepsilon\}$ does not contain a divergent subsequence. Hence $\{x_\varepsilon\}$ contains a convergent subsequence, so we may assume, without loss of generality that $\lim_{\varepsilon \to 0} x_\varepsilon = x_0$ and that $\lim_{\varepsilon \to 0} r_\varepsilon = r_0$.

Therefore we find $M_0 x_0 - y = r_0$ with $x_0 \in \mathscr{R}(M_0)$ and $M_0 r_0 = 0$, or $M_0^* y = x_0$, and hence

$$\lim_{\varepsilon \to 0} M_\varepsilon^* y = \lim_{\varepsilon \to 0} x_\varepsilon = x_0 = M_0^* y,$$

and as we took y arbitrarily from $\mathbb{R}^n$, it follows that $\lim_{\varepsilon\to 0} M_\varepsilon^* = M_0^*$, and we are done with the proof.

We will now state and prove the converse of the previous theorem.

CONVERSE THEOREM. *Let* $\{P_\varepsilon, \varepsilon \in [0,1]\}$ *be such that* $\lim_{\varepsilon\to 0} P_\varepsilon = P_0$ *and such that, with* $M_\varepsilon := I - P_\varepsilon$, *one has* $\lim_{\varepsilon\to 0} M_\varepsilon^* = M_0$. *Then there is an* $\bar{\varepsilon} \in (0,1]$ *such that* $\forall \varepsilon \in [0, \bar{\varepsilon}]$, $\dim \mathcal{N}(M_\varepsilon) = p \leqslant n$.

Proof. Assume to the contrary that there is a sequence $\{\varepsilon' \to 0\}$ with $\dim \mathcal{N}(M_{\varepsilon'}) = \tilde{p}$ and $\dim \mathcal{N}(M_0) > \tilde{p}$. Now consider an arbitrary $y_0 \in \mathcal{N}(M_0)$ and $x_0 = M_0^* y_0$. Then we have $M_0 x_0 - y_0 = r_0$, $M_0 r_0 = 0$, $x_0 \in \mathcal{R}(M_0)$. Hence $M_0^2 x_0 - M_0 y_0 = M^2 x_0 = M_0 r_0 = 0$; hence $x_0 \in \mathcal{N}(M_0^2) = \mathcal{N}(M_0)$ by Lemma 3. Therefore $x_0 = 0$. So $M_0^* y_0 = x_0 = 0$.

Now $M_0^* = \lim_{\varepsilon\to 0} M_\varepsilon^*$. Therefore $M_0^* y_0 = \lim_{\varepsilon\to 0} M_\varepsilon^* y_0$. But $M_\varepsilon^* y_0 := x_\varepsilon$ is equivalent to $M_\varepsilon x_\varepsilon - y_0 \in \mathcal{N}(M_\varepsilon)$ and $x_\varepsilon \in \mathcal{R}(M_\varepsilon)$, or $M_\varepsilon x_\varepsilon - y_0 = r_\varepsilon$, $x_\varepsilon \in \mathcal{R}(M_\varepsilon)$, $M_\varepsilon r_\varepsilon = 0$. We know $\lim_{\varepsilon\to 0} x_\varepsilon = 0$, therefore $\lim_{\varepsilon\to 0} r_\varepsilon$ exists. So we have $y_0 = \lim_{\varepsilon\to 0}(-r_\varepsilon)$, where $M_\varepsilon r_\varepsilon = 0$.

Now take a basis $\{y_0^1, y_0^2, \ldots, y_0^p\} \subseteq \mathcal{N}(M_0)$. Then the previous arguments make clear that for ε sufficiently small, $\dim \mathcal{N}(M_\varepsilon) \geqslant p$, and this contradicts our assumption. Therefore, indeed, we have the conclusion of the converse theorem. ∎

IMPLICATIONS FOR MARKOV CHAINS

We will now see what the implications of the previous results are for $\bar{P}_\varepsilon = I - M_\varepsilon M_\varepsilon^*$, where $\bar{P}_\varepsilon := \lim_{t\to\infty}(I + P_\varepsilon + \cdots + P_\varepsilon^{t-1})/t$ and where $M_\varepsilon = I - P_\varepsilon$.

Suppose that the Markov chain associated with P_ε consists of precisely r ergodic classes, $\forall \varepsilon \in [0,1]$; then $\lim_{\varepsilon\to 0} \bar{P}_\varepsilon = \bar{P}_0$. This can be seen as follows. After possibly a permutation of the states $\{1, 2, \ldots, n\}$ we have

$$P_\varepsilon = \begin{bmatrix} D_1(\varepsilon) & 0 & \cdots & 0 & 0 \\ 0 & D_2(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_r(\varepsilon) & 0 \\ B_1(\varepsilon) & B_2(\varepsilon) & \cdots & B_r(\varepsilon) & C(\varepsilon) \end{bmatrix},$$

where the matrices $D_i(\varepsilon)$ are square and irreducible and $\rho(C(\varepsilon)) < 1$, where $\rho(C(\varepsilon))$ is the spectral radius of $C(\varepsilon)$. We know (see for instance [3, Theorem 1.4, Chapter 2]), that $\rho(D_i(\varepsilon)) = 1$ and $\rho(D_i(\varepsilon))$ is a simple eigenvalue. This implies that rank $I - P_\varepsilon$ is equal to $n - r$ $\forall \varepsilon \in [0, 1]$. Hence the previous results apply and we are done.

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